

Quark mass dependence of the $X(3872)$ binding energy

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Abstract

We explore the quark-mass dependence of the pole position of the $X(3872)$ state within the molecular picture. The calculations are performed within the framework of a nonrelativistic Faddeev-type three-body equation for the $D\bar{D}\pi$ system in the $J^{PC} = 1^{++}$ channel. The πD interaction is parametrised via a D^* pole, and a three-body force is included to render the equations well defined. Its strength is adjusted such that the $X(3872)$ appears as a $D\bar{D}^*$ bound state 0.5 MeV below the neutral threshold. We find that the trajectory of the $X(3872)$ depends strongly on the assumed quark-mass dependence of the short-range interactions which can be determined in future lattice QCD calculations. At the same time we are able to provide nontrivial information on the chiral extrapolation in the X channel.

Keywords: exotic hadrons, charmonium, chiral dynamics, effective field theory

1. Introduction

A decade ago, a new era in hadronic spectroscopy began with the observation by the Belle Collaboration of the charmonium-like meson $X(3872)$ [1] which was quickly confirmed by many other collaborations, see Ref. [2] for a recent review article. In fact, such a state was predicted to exist as a $D\bar{D}^*$ bound system, analogous to the deuteron, long before its discovery, based on scattering calculations using a one-pion-exchange (OPE) potential [3, 4]. This state possesses a number of intriguing features which make it an attractive object for both experimental and theoretical studies. In particular, it is seen with approximately equal branching fractions in the modes $\pi^+\pi^-J/\psi$ [5] and $\pi^+\pi^-\pi^0J/\psi$ [6], which indicates a strong isospin violation in the X decay. Recently, the quantum numbers of the $X(3872)$ were measured to be 1^{++} [7], consistent with its being an S -wave $D\bar{D}^{*1}$ bound system [8–11]. This interpretation finds support especially in the very small binding energy, $E_B < 1$ MeV, with respect to the $D^0\bar{D}^{*0}$ channel [12].

¹This shorthand notation is used for the proper C -parity eigenstate.

While some authors claim that the iterated OPE potential alone is strong enough to form the $X(3872)$ [13–16], others come to an opposite conclusion, see, for example, [17, 18]. Different (complementary or alternative to the OPE) short-range $D\bar{D}^*$ mechanisms in the X are considered, for instance, in [19–22]. It has to be noticed, however, that pions are treated as static in most models for the X involving OPE. Furthermore, as stressed in [23], since the X resides only 7 MeV above the $D\bar{D}\pi$ threshold, the presence of the corresponding cut might weaken the pion potential considerably.

In order to better understand the role of the OPE interaction including that of the three-body cut as well as to use a treatment with minimal bias, the $X(3872)$ was studied in [24] within a three-body scattering Faddeev-type formalism for the $D\bar{D}\pi$ system. The πD interactions were parametrised via the D^* pole. In addition to the OPE, a $D\bar{D}^*$ contact term with a strength C_0 was also included to parametrise unknown short-distance physics. The resulting interaction was iterated to all orders through the solution of the integral equation. The equations were solved in momentum space using a sharp cut-off prescription. The scale-dependence of the contact term was determined by requiring the equations to have a pole at a fixed binding energy of 0.5 MeV for a wide range of different cut-offs Λ . It was found, in particular, that C_0 varies strongly with Λ taking values between $\pm\infty$. Interestingly, for $\Lambda \simeq 1$ GeV, the contact term turns out to be very small which might explain the results of [13–16]. However, it follows from [24] that these findings are scheme-dependent. The low-energy dynamics of the $D\bar{D}^*$ system was also investigated within a pionless EFT [25] as well as the so-called X-EFT approach (see, for example, [26, 27]) that assumes that pions can be treated perturbatively.

In this paper we apply the formalism developed in [24] to explore the behaviour of the binding energy E_B as the quark masses go away from their physical value. Since the pion mass enters our calculations explicitly and the pion mass squared is directly proportional to the quark mass then, equivalently, in the following we talk about the pion mass dependence of E_B . To this end, we treat all masses and coupling constants as functions of the pion mass m_π and define their physical values (labelled as “ph”) as those which correspond to the physical pion mass m_π^{ph} or, equivalently, the physical values of the light quark masses. We refer to this case as to the physical limit or physical point. We then perform an expansion of all such quantities in terms of the parameter $\delta m_\pi/M$, where the small scale is $\delta m_\pi = m_\pi - m_\pi^{\text{ph}}$, while the large scale M is given by a typical hadronic scale such as, for instance, the D - or D^* -meson mass or the chiral symmetry breaking scale $\sim 4\pi f_\pi$, with f_π the pion decay constant. In either case, M appears to be of order 1 GeV so that keeping both the leading term in the expansion $\propto \delta m_\pi^2/M^2$ and the leading term in the chiral expansion of the $D\bar{D}^*$ potential, as formulated above, is justified for $\delta m_\pi \sim m_\pi^{\text{ph}}$. We assume that this framework might be still applicable in the case of larger pion masses of the order of 300–400 MeV, which is the typical value used in today’s lattice QCD calculations in the charm sector [28].

It should be stressed that in the approach used here, besides the known m_π -dependences of the masses and coupling constants, we also need to include a dependence of the short-range interaction C_0 with an *a priori* unknown strength, in analogy to the investigations in the

nucleon–nucleon (NN) sector [29, 30]². The inclusion of m_π -dependence of the short-range interaction is necessary not only because of power counting arguments, but also demanded by the renormalisation group: a Λ - and m_π -dependent short-ranged operator is needed to absorb the Λ -dependence we encounter when varying the pion mass in the other quantities and especially in the pion propagator. Our formalism seems more complicated than that of [16], where it is claimed that a model-independent, parameter-free prediction can be given for the pion mass dependence of the X . However, it is our understanding that the result of [16] is regularisation scheme as well as scale-dependent.

We find that, as expected, the pion exchange itself gets weaker with increasing pion mass. Meanwhile, the m_π -dependence of the counter term can now either enhance this feature, thus leading to a rapid disappearance of the $X(3872)$ pole as m_π is increased, or, on the contrary, weaken the effect even that much, that the X binding increases with increasing pion mass. In any case, the pion mass dependence $E_B(m_\pi)$ turns out to be nontrivial. Thus our work provides important insights for the chiral extrapolation in the $X(3872)$ sector. Furthermore, with the help of our findings and from the pion mass dependence of the $X(3872)$ pole once it is available from lattice simulations, we will eventually be able to extract valuable information on the physics of the interaction that leads to the formation of the $X(3872)$.

2. Scattering equations

In this section we outline briefly our theoretical formulation following closely the lines of [24]. The lowest-order $D^*D\pi$ interaction Lagrangian has the form [26]

$$\mathcal{L} = \frac{g_c}{\sqrt{2}f_\pi} \left(\mathbf{D}^{*\dagger} \cdot \nabla \pi^a \tau^a D + D^\dagger \tau^a \nabla \pi^a \cdot \mathbf{D}^* \right), \quad \pi = \begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix}, \quad (1)$$

where g_c is the dimensionless $D^*D\pi$ coupling constant. The latter can be fixed from the $D^{*0} \rightarrow D^0\pi^0$ width via the relation

$$\Gamma(D^{*0} \rightarrow D^0\pi^0) \equiv \Gamma_* = \frac{g_c^2 m_0}{24\pi f_\pi^2 m_{*0}} [2\mu_q(D^0\pi^0)(m_{*0} - m_0 - m_{\pi^0})]^{3/2}, \quad (2)$$

where the reduced mass is defined as $\mu_q(XY) = m_X m_Y / (m_X + m_Y)$. Here and in what follows, m_0 , m_c , m_{*0} , m_{*c} , m_{π^0} , and m_{π^c} are the masses of the neutral and charged D mesons, D^* mesons, and pions, respectively.

In the physical limit one has [12] $f_\pi^{\text{ph}} = 92.4$ MeV,

$$\begin{aligned} m_{\pi^0}^{\text{ph}} &= 134.98 \text{ MeV}, & m_0^{\text{ph}} &= 1864.84 \text{ MeV}, & m_{*0}^{\text{ph}} &= 2006.97 \text{ MeV}, \\ m_{\pi^c}^{\text{ph}} &= 139.57 \text{ MeV}, & m_c^{\text{ph}} &= 1869.62 \text{ MeV}, & m_{*c}^{\text{ph}} &= 2010.27 \text{ MeV}, \end{aligned}$$

²See also a recent calculation of [31] where the pion mass dependence of NN contact interactions was determined by resonance saturation using unitarised chiral perturbation theory combined with lattice QCD results.

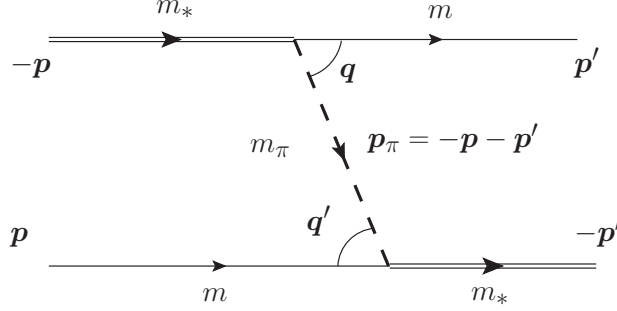


Figure 1: Kinematics of the $D\bar{D}^*$ scattering due to the OPE. Double lines denote D^* 's, single lines denote D 's, while the dashed line stands for the pion.

while the value $\Gamma_*^{\text{ph}} = 42$ keV can be deduced from the data for the charged D^* decay modes [12]. Then relation (2) gives $g_c^{\text{ph}} = 0.62$.

Using the notation of [24], the P -wave $D^*D\pi$ vertex can be written as

$$v_{D^*D\pi}(\mathbf{q}) = g \boldsymbol{\epsilon} \cdot \mathbf{q}, \quad (3)$$

where $\boldsymbol{\epsilon}$ is the D^* polarisation vector and \mathbf{q} is the relative momentum in the $D\pi$ system. The vertex coupling g is related to the dimensionless constant g_c from the Lagrangian (1) as

$$g = \frac{g_c}{(4\pi)^{3/2} f_\pi} \left(\frac{m_0}{m_{*0} \mu_q(D^0\pi^0)} \right)^{1/2}, \quad (4)$$

and its physical value is [24]

$$g^{\text{ph}} = 1.29 \cdot 10^{-5} \text{ MeV}^{-3/2}. \quad (5)$$

The OPE potential visualised in Fig. 1 is

$$V^{nn'}(\mathbf{p}, \mathbf{p}') = -g^2 \frac{(\mathbf{p}' + \alpha \mathbf{p})_n (\mathbf{p} + \alpha \mathbf{p}')_{n'}}{D_3(\mathbf{p}, \mathbf{p}')}, \quad \alpha = \frac{m}{m + m_\pi}, \quad (6)$$

where the indices n, n' are contracted with the corresponding indices of the D^* polarisation vectors. The inverse three-body propagator reads

$$D_3(\mathbf{p}, \mathbf{p}') = 2m + m_\pi + \frac{p^2}{2m} + \frac{p'^2}{2m} + \frac{(\mathbf{p} + \mathbf{p}')^2}{2m_\pi} - M - i0. \quad (7)$$

The OPE potential (6) interrelates the four D -meson channels defined as

$$|0\rangle = D^0 \bar{D}^{*0}, \quad |\bar{0}\rangle = \bar{D}^0 D^{*0}, \quad |c\rangle = D^+ D^{*-}, \quad |\bar{c}\rangle = D^- D^{*+}. \quad (8)$$

For the sake of convenience, we define the energy E relative to the neutral two-body threshold:

$$M = m_{*0} + m_0 + E. \quad (9)$$

Given the above definitions, the system of coupled Lippmann-Schwinger equations for the $D\bar{D}^*$ t -matrix elements $a_{00}^{nn'}(\mathbf{p}, \mathbf{p}')$ and $a_{c0}^{nn'}(\mathbf{p}, \mathbf{p}')$ in the C -even channel has the form:

$$\begin{aligned} a_{00}^{nn'}(\mathbf{p}, \mathbf{p}', E) &= \lambda_0 V_{00}^{nn'}(\mathbf{p}, \mathbf{p}') - \sum_{i=0,c} \lambda_i \int \frac{d^3k}{\Delta_i(k)} V_{0i}^{nm}(\mathbf{p}, \mathbf{k}) a_{i0}^{mn'}(\mathbf{k}, \mathbf{p}', E), \\ a_{c0}^{nn'}(\mathbf{p}, \mathbf{p}', E) &= \lambda_c V_{c0}^{nn'}(\mathbf{p}, \mathbf{p}') - \sum_{i=0,c} \lambda_i \int \frac{d^3k}{\Delta_i(k)} V_{ci}^{nm}(\mathbf{p}, \mathbf{k}) a_{i0}^{mn'}(\mathbf{k}, \mathbf{p}', E), \end{aligned} \quad (10)$$

where $\lambda_0 = \langle 0 | \vec{\tau}_1 \cdot \vec{\tau}_2 | \bar{0} \rangle = \langle c | \vec{\tau}_1 \cdot \vec{\tau}_2 | \bar{c} \rangle = 1$ and $\lambda_c = \langle 0 | \vec{\tau}_1 \cdot \vec{\tau}_2 | \bar{c} \rangle = \langle c | \vec{\tau}_1 \cdot \vec{\tau}_2 | \bar{0} \rangle = 2$ are isospin factors for the π^0 - and π^\pm -exchange, respectively. The inverse two-body propagators Δ_0 and Δ_c take the form

$$\Delta_0(p) = \frac{p^2}{2\mu_0} - E - \frac{i}{2}\Gamma_0(p), \quad \Delta_c(p) = \frac{p^2}{2\mu_c} + \delta - E - \frac{i}{2}\Gamma_c(p), \quad (11)$$

where $\delta = m_{*c} + m_c - m_{*0} - m_0$ and the “running” widths are given by

$$\Gamma_0(p) = \Gamma(D^{*0} \rightarrow D^0 \gamma) + \frac{8\pi^2}{3} g^2 \sum_{k=0,c} \lambda_k \mu_{qkk} (q_{0kk}^3 + i\tilde{\kappa}_{0kk}^3) \quad (12)$$

$$\Gamma_c(p) = \frac{8\pi^2}{3} g^2 \sum_{j,k=0,c \text{ \& } j \neq k} \lambda_k \mu_{qjk} (q_{cjk}^3 + i\tilde{\kappa}_{cjk}^3). \quad (13)$$

Here, $\mu_{qjk} \equiv \mu_q(D^j \pi^k)$ and the momentum

$$q_{ijk} \equiv \sqrt{2\mu_{qjk} \left(M - m_i - m_j - m_{\pi^k} - \frac{p^2}{2\mu_{*i}} \right)}, \quad i, j, k = 0, c, \quad (14)$$

continues analytically below the corresponding threshold as $q_{ijk} \rightarrow i\kappa_{ijk}$. The constant shifts

$$\tilde{\kappa}_{ijk} = \sqrt{2\mu_{qjk}(m_j + m_{\pi^k} - m_{*i})} \Theta(m_j + m_{\pi^k} - m_{*i}) \quad (15)$$

cancel the contributions of the κ_{ijk} 's at thresholds and, therefore, guarantee that the neutral and charged D^* masses are defined as zeros of the real part of the inverse propagators (11). This implies that, as the pion mass m_π deviates from its physical value m_π^{ph} , the D^* masses acquire contributions of the form:

$$\begin{aligned} \Delta m_{*0} &= \frac{1}{2} \text{Im} [\Gamma_0(p, m_\pi^{\text{ph}}) - \Gamma_0(p, m_\pi)]|_{p=0, E=0}, \\ \Delta m_{*c} &= \frac{1}{2} \text{Im} [\Gamma_c(p, m_\pi^{\text{ph}}) - \Gamma_c(p, m_\pi)]|_{p=0, E=\delta}. \end{aligned} \quad (16)$$

The amplitudes $a_{ik}^{nn'}(\mathbf{p}, \mathbf{p}', E)$ can be represented as

$$a_{ik}^{nn'}(\mathbf{p}, \mathbf{p}', E) = a_{ik}^{SS}(p, p', E) T_{SS}^{nn'} + a_{ik}^{DS}(p, p', E) T_{DS}^{nn'} \quad (17)$$

with the help of the S - and D -wave projectors

$$T_{SS}^{nn'} = \frac{1}{4\pi} \delta_{nn'}, \quad T_{DS}^{nn'} = \frac{1}{4\pi\sqrt{2}} (\delta_{nn'} - 3n_n n_{n'}). \quad (18)$$

For the quantum numbers 1^{++} and for the neutral mesons in the final state, we are interested only in the a_{00}^{SS} . A similar decomposition is done for the potential $V_{ik}^{nn'}$. Then, in order to describe the short-range dynamics, we modify the component of the potential V_{ik}^{SS} as

$$V_{ik}^{SS}(p, p') \rightarrow C_0 + V_{ik}^{SS}(p, p'), \quad (19)$$

where the constant contact interaction C_0 is tuned to produce a bound state at $E = -E_B = -0.5$ MeV. It has to be noticed that, because of the P -wave vertices, the integrals in the Lippmann–Schwinger equation (10) diverge. We regularise them by a sharp cut-off Λ and assume a proper Λ -dependence of the contact interaction $C_0(\Lambda)$ which ensures that physical observables are (approximately) Λ -independent. This program was successfully carried out in [24] where all further details can be found. We only stress here that our approach explicitly preserves unitarity and treats OPE nonperturbatively.

3. Masses, decay constants, and widths

For the sake of convenience, we introduce the ratio $m_\pi/m_\pi^{\text{ph}} \equiv \xi$ and describe the “running” behaviour of all physical quantities entering the problem in terms of this ratio ξ . Physical values are readily restored for $\xi = 1$.

The m_π -dependence of the D - and D^* -meson masses was studied in a recent paper [32] within the framework of unitarised $SU(3)$ chiral perturbation theory. The tree-level expressions for $m(\xi)$ and $m_*(\xi)$ corresponding to the expansion about the $SU(2)$ chiral limit with the charm quark mass fixed at its physical value have the form

$$m(\xi) = m^{\text{ph}} \left[1 + h_1 \left(\frac{m_\pi^{\text{ph}}}{m^{\text{ph}}} \right)^2 (\xi^2 - 1) \right], \quad m_*(\xi) = m_*^{\text{ph}} \left[1 + h_1 \left(\frac{m_\pi^{\text{ph}}}{m_*^{\text{ph}}} \right)^2 (\xi^2 - 1) \right] + \Delta m_*, \quad (20)$$

where $h_1 \approx 0.42$, the shift Δm_* is given in (16) and, in order to simplify the presentation, we do not distinguish between neutral and charged states.

For the pion decay constant, we resort to the one-loop chiral perturbation theory result (see, for example, [33]) which in our notation has the form

$$f_\pi(\xi) = f_\pi^{\text{ph}} \left[1 + \left(1 - \frac{f}{f_\pi^{\text{ph}}} \right) (\xi^2 - 1) - \frac{(m_\pi^{\text{ph}})^2}{8\pi^2 f^2} \xi^2 \ln \xi \right]. \quad (21)$$

Notice that we have expressed the corresponding low-energy constant l_4 in terms of the pion decay constant f in the chiral limit for which we adopt the value [34]

$$f_\pi(m_\pi = 0) \equiv f = 85 \text{ MeV}. \quad (22)$$

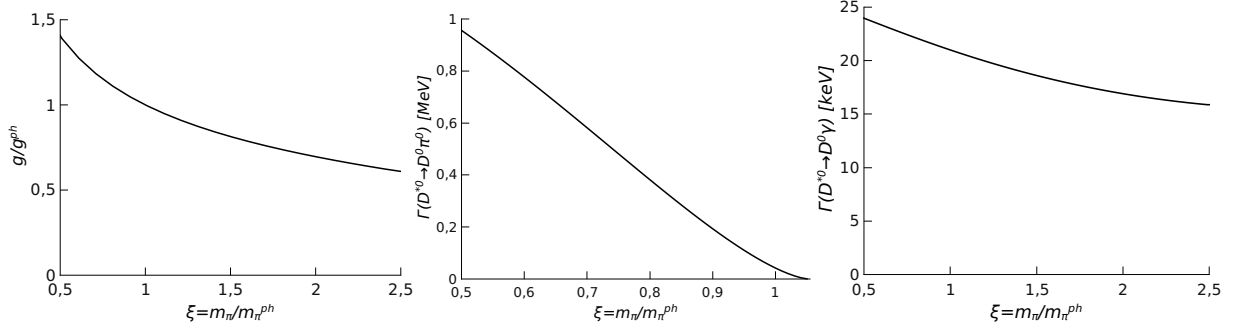


Figure 2: Dependence of the ratio g/g^{ph} (left panel), the $D^{*0} \rightarrow D^0 \pi^0$ width (middle panel) and the $D^{*0} \rightarrow D^0 \gamma$ width (right panel) on the varying pion mass $\xi = m_\pi/m_\pi^{\text{ph}}$.

To establish the m_π -dependence of the $D^* D \pi$ coupling g we use the lattice results of [34] (in particular, the fit ChPT-II which incorporates one-loop effects). In order to avoid confusion with notations, we use the subscript “lat” for some parameters from [34]. We also re-define the pion decay constant of [34] to comply with the standard convention (see, in particular, (22)). As follows from (4),

$$g = g^{\text{ph}} \left(\frac{m_\pi^{\text{ph}}}{m_\pi} \right)^{1/2} \left(\frac{m_*^{\text{ph}}}{m_*} \right)^{1/2} \left(\frac{m + m_\pi}{m^{\text{ph}} + m_\pi^{\text{ph}}} \right)^{1/2} \left(\frac{f_\pi^{\text{ph}}}{f_\pi} \right) \left(\frac{g_c}{g_c^{\text{ph}}} \right), \quad (23)$$

with the ratio g_c/g_c^{ph} as a function of ξ extracted from [34] in the form:

$$\frac{g_c}{g_c^{\text{ph}}} = 1 + C_1(\xi^2 - 1) + C_2 \xi^2 \ln \xi, \quad (24)$$

$$C_1 = 1 - \left(1 - \frac{(1 + 2g_0^2)}{8\pi^2} \left(\frac{m_\pi^{\text{ph}}}{f} \right)^2 \ln \frac{m_\pi^{\text{ph}}}{\mu_{\text{lat}}} + \alpha_{\text{lat}} (m_\pi^{\text{ph}})^2 \right)^{-1}, \quad (25)$$

$$C_2 = -\frac{(1 + 2g_0^2)}{8\pi^2} \left(\frac{m_\pi^{\text{ph}}}{f} \right)^2 \left(1 - \frac{(1 + 2g_0^2)}{8\pi^2} \left(\frac{m_\pi^{\text{ph}}}{f} \right)^2 \ln \frac{m_\pi^{\text{ph}}}{\mu_{\text{lat}}} + \alpha_{\text{lat}} (m_\pi^{\text{ph}})^2 \right)^{-1}. \quad (26)$$

Further, the values of the various parameters taken from [34] are:

$$g_0 = 0.46, \quad \alpha_{\text{lat}} = -0.16 \text{ GeV}^{-2}, \quad \mu_{\text{lat}} = 1 \text{ GeV}. \quad (27)$$

In Fig. 2 we give the m_π -dependence of the coupling constant g (left panel) and that of the $D^{*0} \rightarrow D^0 \pi^0$ decay width (middle panel). The width $\Gamma(D^{*0} \rightarrow D^0 \pi^0)$ vanishes already at $m_\pi/m_\pi^{\text{ph}} \approx 1.05$ because of the closing phase space. The coupling constant g does not vanish at this point, however, and defines the analytical continuation of the loop operator below threshold.

For the $D^* \rightarrow D \gamma$ width, we adopt the results derived in [35], namely

$$\Gamma(D^{*0} \rightarrow D^0 \gamma) \equiv \Gamma_\gamma = \frac{1}{3} \alpha_{\text{EM}} |\mu_1|^2 k_\gamma^3, \quad (28)$$

where $\alpha_{\text{EM}} = 1/137$ is the fine structure constant, k_γ is the photon momentum, and μ_1 is the transition magnetic moment which depends on the pion mass. Notice that in [35], in addition to pion loops, also effects of the kaon loops were considered in order to determine the quark mass dependence of μ_1 . In the framework of SU(2) chiral perturbation theory, any m_π -dependence emerging through the kaon mass is represented by local, m_π^2 -dependent operators in the effective Lagrangian. At the order in the ξ -expansion we are working, the contributions from the kaon loops to the m_π -dependence of μ_1 can be neglected. The resulting dependence of Γ_γ on m_π is visualised in the right panel of Fig. 2, while the physical value deduced from the charged D^* decays modes is $\Gamma_\gamma^{\text{ph}} = 21$ keV. Taking into account the effects of kaon loops from [35] leads to a sort of resonance saturation of the corresponding higher-order terms that gives rise to an extra m_π -dependence of Γ_γ . We have verified that the resulting dependence $\Gamma_\gamma(\xi)$ is nearly indistinguishable from the one shown in Fig. 2.

4. The contact interaction

In this section we discuss in detail the derivation of the dependence of the contact interaction strength, C_0 , on the pion mass. As was explained above, this term incorporates the unknown short-range dynamics present in the system. The nature of the latter part is difficult to determine, so we are forced to resort to an educated guess on the relevant dependence $C_0(m_\pi)$ based on (i) the assumption that the X is a bound state and (ii) the requirement that the binding energy $E_B(m_\pi)$ is Λ -independent (that is, one is able to renormalise the full theory with dynamical pions at least in the leading order in Λ). Thus we postulate for the “running” contact interaction the form

$$C_0(\Lambda, \xi) = C_0^{\text{ph}} + \delta C_0 = C_0^{\text{ph}}(\Lambda) \left(1 + f(\Lambda) \frac{\delta m_\pi^2}{M^2} \right) = C_0^{\text{ph}}(\Lambda) \left(1 + f(\Lambda) \frac{(m_\pi^{\text{ph}})^2}{M^2} (\xi^2 - 1) \right), \quad (29)$$

where, as it was explained in the introduction, the expansion is done in terms of the ratio of the small scale δm_π to the large scale M , and only the leading term is kept. The leading Λ -dependence of the contact interaction is captured by the physical-limit quantity $C_0^{\text{ph}}(\Lambda)$, while the dimensionless function $f(\Lambda)$ absorbs the extra Λ -dependence which appears for values of the pion mass away from the physical point. Therefore, we fix the Λ -dependence of the contact interaction requiring that both the binding energy E_B itself as well as its slope at the physical point, $(\partial E_B / \partial m_\pi)|_{m_\pi=m_\pi^{\text{ph}}}$, are Λ -independent.

Here and in what follows we adopt the value of $M \sim m_\rho \sim 800$ MeV for the relevant hard scale in the problem with dynamical pions. Further, since Λ has to be larger than the maximal pion mass considered in the study, $m_\pi \simeq 300$ MeV, but smaller than the hard scale M , we let the cut-off Λ vary in the range of $400 \dots 700$ MeV.

In order to get a better insight for the $C_0(\Lambda, \xi)$ and, in particular, investigate properties of the function $f(\Lambda)$, we first consider an analytically solvable model with the interaction given by the contact potential C_0 only [24]. We also stick to the isospin limit for simplicity. We start from the physical pion mass. The system of Lippmann–Schwinger equations (10) reduces to two decoupled equations for the amplitudes $a_0 = (a_{00} - a_{c0})/2$ and

$a_1 = (a_{00} + a_{c0})/2$ which correspond to the isosinglet and isotriplet channels, respectively. In the isoscalar channel relevant for the X , the equation reads:

$$a_0 = -C_0 + C_0 I(E) a_0, \quad (30)$$

where

$$I(E) = \int_0^\Lambda ds \frac{s^2}{s^2/(2\mu) - E} \approx 2\mu\Lambda - \pi\mu\sqrt{-2\mu E} + O\left(\frac{\sqrt{-2\mu E}}{\Lambda}\right), \quad \mu = \frac{mm_*}{m + m_*}. \quad (31)$$

The system possesses a bound state in the isosinglet channel at $E = -E_B$ if

$$C_0 I(-E_B) = 1, \quad (32)$$

as follows from (30). This means that while both C_0 and $I(-E_B)$ are strongly Λ -dependent, their product is a renormalisation group invariant Λ -independent quantity, and the bound-state equation (32) defines the contact interaction to be $C_0(\Lambda) = I^{-1}(-E_B, \Lambda)$.

If the pion mass deviates from m_π^{ph} , the corresponding expansion of C_0 near its physical value reads:

$$\begin{aligned} C_0 = I^{-1}(-E_B) &\approx I^{-1}(-E_B^{\text{ph}}) + \frac{\partial I^{-1}(-E_B^{\text{ph}})}{\partial \mu^{\text{ph}}} \frac{\partial \mu^{\text{ph}}}{\partial m_\pi^2} \delta m_\pi^2 + \frac{\partial I^{-1}(-E_B^{\text{ph}})}{\partial E_B^{\text{ph}}} \frac{\partial E_B^{\text{ph}}}{\partial m_\pi^2} \delta m_\pi^2 \\ &= C_0^{\text{ph}} \left[1 - \left(\frac{\partial \ln I(-E_B^{\text{ph}})}{\partial \mu^{\text{ph}}} \frac{\partial \mu^{\text{ph}}}{\partial m_\pi^2} + \frac{\partial \ln I(-E_B^{\text{ph}})}{\partial E_B^{\text{ph}}} \frac{\partial E_B^{\text{ph}}}{\partial m_\pi^2} \right) \delta m_\pi^2 \right]. \end{aligned} \quad (33)$$

It can be re-written in the form of (29) with

$$\frac{f(\Lambda)}{M^2} = -\frac{h_1}{(m^{\text{ph}})^2} + \frac{\pi\mu^{\text{ph}}}{2\Lambda\sqrt{2\mu^{\text{ph}}E_B^{\text{ph}}}} \frac{\partial E_B^{\text{ph}}}{\partial m_\pi^2}, \quad (34)$$

where we used that

$$\frac{\partial \ln I(-E_B)}{\partial E_B} \approx -\frac{\pi\mu}{2\Lambda\sqrt{2\mu E_B}}, \quad \frac{\partial \ln I(-E_B)}{\partial \mu} \frac{\partial \mu}{\partial m_\pi^2} = h_1 \left(\frac{1}{m^2} - \frac{1}{mm_*} + \frac{1}{m_*^2} \right) \approx \frac{h_1}{m^2}, \quad (35)$$

and the difference between the m_* and m is neglected. The constant h_1 is given at the beginning of Sect. 3. Equation (34) yields an analytical expression for the variation of the coefficient f with the cut-off Λ which keeps the slope $(\partial E_B/\partial m_\pi)|_{m_\pi=m_\pi^{\text{ph}}}$ in the contact theory independent of Λ . In the theory with dynamical pions, the same requirement that the binding energy slope at the physical point is Λ -independent allows one to determine the cut-off dependence $f(\Lambda)$ numerically.

Finally, in order to implement (29) in the calculation of the binding energy E_B as a function of the pion mass m_π , we need an estimate for $f(\Lambda)$. To this end we consider an example of a physical mechanism which is not included explicitly in our calculations and

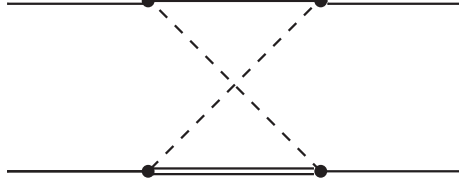


Figure 3: Two-pion exchange contribution. Double (single) lines denote D^* 's (D 's) while dashed lines denote pions.

estimate its contribution to the m_π -dependence of C_0 . Specifically, we consider the crossed-box two-pion-exchange (TPE) diagram shown in Fig. 3. The corresponding amplitude is $M_{\text{TPE}} \sim (g_c \sqrt{mm_*}/f_\pi)^4 \times I_{\text{TPE}}$, with

$$I_{\text{TPE}} = \int \frac{d^4 q}{(2\pi)^4} \frac{q^4}{(q^2 - m_\pi^2 + i0)^2 ((p - q)^2 - m_*^2 + i0) ((p_* - q)^2 - m^2 + i0)}, \quad (36)$$

where $p = (m, \mathbf{0})$ and $p_* = (m_*, \mathbf{0})$ are the 4-momenta of the initial (and final) D and D^* meson, respectively. We now estimate the contribution of the TPE diagram to $f(\Lambda)$ with the help of a naive dimensional analysis (NDA). This can be most easily achieved in the framework of time-ordered perturbation theory:

$$I_{\text{TPE}} \approx -i \int \frac{d^3 q}{(2\pi)^3} \frac{1}{(2\omega)(2\omega_*)(2\omega_\pi)^2} \frac{q^4}{(m + m_* - 2\omega_\pi - \omega - \omega_*)(m_* - \omega - \omega_\pi + i0)^2}, \quad (37)$$

where $\omega_i = \sqrt{\mathbf{q}^2 + m_i^2}$ are the energies of the π -, D -, and D^* -meson and we keep only the dominant time ordering with two πDD and one $\pi\pi DD^*$ intermediate states. There are, in general, three scales contributing to the integral: (i) $q \sim \Lambda$, (ii) $q \sim m_\pi$, and (iii) $q \sim \sqrt{2m_\pi(m_* - m - m_\pi)}$ (due to the πDD cuts). The relevant short-range contribution $\propto m_\pi^2$ stems only from the first scale after expanding the integral to the order m_π^2/Λ^2 and from the second scale, both yielding $m_\pi^2/[(4\pi)^2 mm_*]$ for the relevant part of the integral.

Thus, relative to C_0^{ph} , the m_π -dependent part of the short-range interaction δC_0 induced by the TPE contribution can be estimated as

$$\frac{\delta C_0}{C_0^{\text{ph}}} \sim \frac{M_{\text{TPE}}}{M_{\text{OPE}}} \sim g_c^2 \frac{\delta m_\pi^2}{(4\pi f_\pi)^2} \sim \left(g_c^2 \frac{M^2}{(4\pi f_\pi)^2} \right) \frac{\delta m_\pi^2}{M^2}, \quad (38)$$

where we used the fact that the contact term C_0^{ph} is saturated by the OPE amplitude at large momenta, $M_{\text{OPE}} \sim (g_c \sqrt{mm_*}/f_\pi)^2$. Matching (38) and (29), with $M \sim 4\pi f_\pi$ and $g_c \sim 1$, gives $f(\Lambda) \sim 1$.

The estimate for $f(\Lambda)$ sets a constraint on the magnitude of the slope of the binding energy (see, for example, (34) for the case of the contact theory). In particular, for $f(\Lambda) \sim 1$ the shift of the binding energy $\delta E_B \sim E_B$ for $\delta m_\pi \sim m_\pi$ can be interpreted as natural. In Fig. 4, as dotted curves, we show the trajectories $f(\Lambda)$ which correspond to the (cut-off-independent) slope of $(\partial E_B / \partial m_\pi)|_{m_\pi = m_\pi^{\text{ph}}} = \pm E_B^{\text{ph}} / m_\pi^{\text{ph}} = \pm 0.4 \times 10^{-2}$. One can see that, at

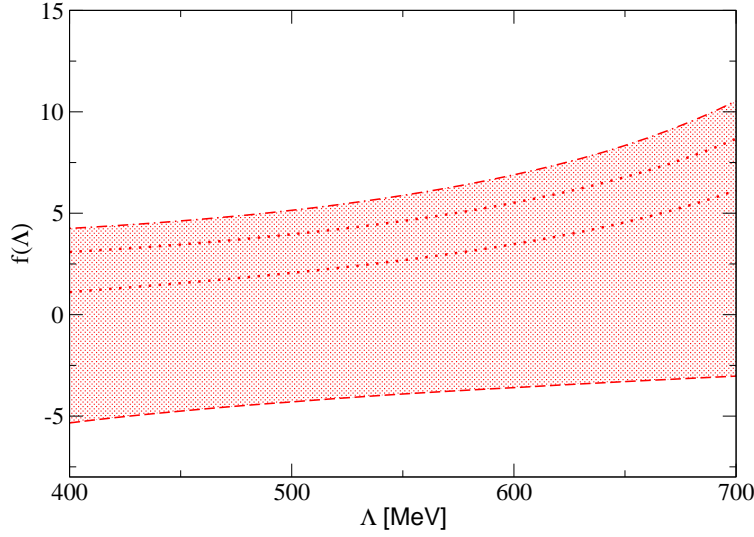


Figure 4: Cut-off dependence of the quantity $f(\Lambda)$ for various values of the slope $(\partial E_B/\partial m_\pi)|_{m_\pi=m_\pi^{\text{ph}}}$. The red filled band indicates the values of $f(\Lambda)$ regarded as natural. The narrow band bounded by the dotted lines corresponds to the natural range for $f(\Lambda)$ derived from the contact theory.

these trajectories $f(\Lambda) \sim 1$, so that, indeed, the estimates $f(\Lambda) \sim 1$ and $\delta E_B/\delta m_\pi \sim E_B/m_\pi$ are reasonably consistent with each other.

Given that other mechanisms, not considered here, can potentially affect $f(\Lambda)$ too, we use a more conservative estimate for $f(\Lambda)$ and regard as natural its values lying in the range $-5 \dots +5$. This leads to the broad filled band shown in Fig. 4. Then, as the pion mass varies in the range $m_\pi^{\text{ph}} \leq m_\pi \leq 2m_\pi^{\text{ph}}$, the change in the contact term δC_0 in (29) does not exceed 50% of C_0^{ph} .

To summarise, in the full problem which incorporates dynamical pions, we treat as natural the dependence of the contact interaction on m_π as given by (29) and adopt a conservative estimate for $f(\Lambda)$, as shown by the filled red band in Fig. 4. The explicit form of $f(\Lambda)$ is determined by demanding the slope $(\partial E_B/\partial m_\pi)|_{m_\pi=m_\pi^{\text{ph}}}$ to be independent of Λ , the latter requirement to be understood as a numerical implementation of the renormalisation group.

5. Results and discussion

As explained in the previous section, the functional form of the m_π -dependent short-range term $f(\Lambda)$ is determined by the requirement that the slope $(\partial E_B/\partial m_\pi)|_{m_\pi=m_\pi^{\text{ph}}}$ is cut-off-independent. Based on our conservative estimate, we allow for those values of $f(\Lambda)$ that, for $400 \text{ MeV} \leq \Lambda \leq 700 \text{ MeV}$, lie within the red filled band in Fig. 4. The corresponding allowed values for the slope are $(\partial E_B/\partial m_\pi)|_{m_\pi=m_\pi^{\text{ph}}} = (-2.2 \dots 0.7) \times 10^{-2}$, where the lowest value corresponds to the dashed curve in Fig. 4 (lower bound) whereas the largest one corresponds to the dot-dashed curve (upper bound). Results for the pion mass dependence of the $X(3872)$ binding energy are shown in Fig. 5. As the pion mass increases, the pion

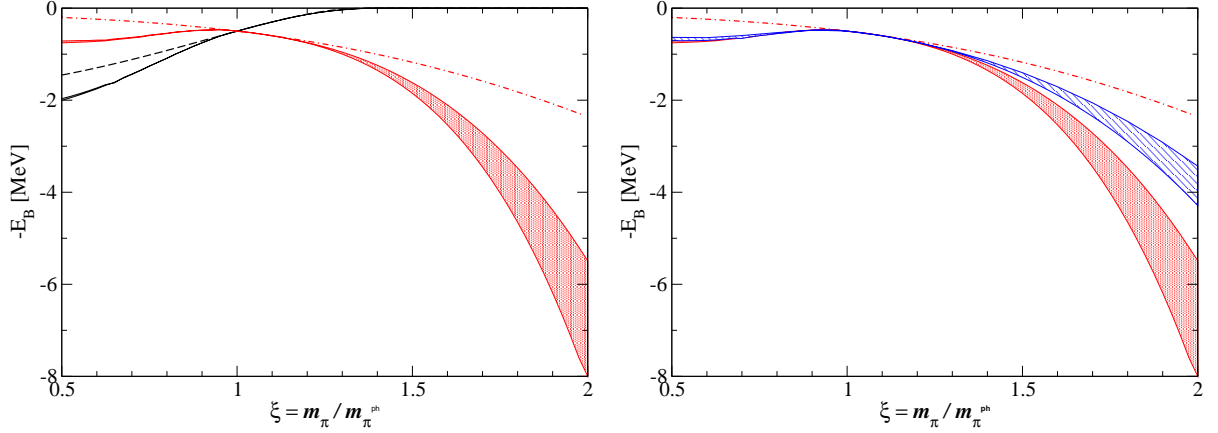


Figure 5: The pion mass dependence of the binding energy for various values of the slope $(\partial E_B / \partial m_\pi)|_{m_\pi=m_\pi^{\text{ph}}}$ and for different cut-offs. Left panel: the red filled band corresponds to $(\partial E_B / \partial m_\pi)|_{m_\pi=m_\pi^{\text{ph}}} = 0.7 \times 10^{-2}$ (the upper bound of the allowed $f(\Lambda)$ shown by the dot-dashed line in Fig. 4) for the cut-off variation in the range $\Lambda \in [400, 700]$ MeV; the black filled band (that looks almost like a single curve) is the same but for the negative slope $(\partial E_B / \partial m_\pi)|_{m_\pi=m_\pi^{\text{ph}}} = -1.5 \times 10^{-2}$. Right panel: the pion mass dependence of the binding energy for the slope $(\partial E_B / \partial m_\pi)|_{m_\pi=m_\pi^{\text{ph}}} = 0.7 \times 10^{-2}$ and for different cut-offs, namely, red filled band for $\Lambda \in [400, 700]$ MeV, as in the left panel; blue hatched band for unnaturally low cut-offs $\Lambda \in [150, 250]$ MeV. The dashed and dash-dotted curves in both panels correspond to the results of the pure contact theory.

exchange gets weaker. Therefore, in the absence of additional short-range forces ($f(\Lambda) \equiv 0$), the binding energy would smoothly decrease with the increase of the pion mass.

For negative values of $f(\Lambda)$, the contact interaction interferes constructively with the OPE contribution resulting in a more rapid fall of E_B . This scenario is exemplified in the left panel of Fig. 5 by the black band corresponding to $(\partial E_B / \partial m_\pi)|_{m_\pi=m_\pi^{\text{ph}}} = -1.5 \times 10^{-2}$ and $\Lambda \in [400, 700]$ MeV. The larger is $|f(\Lambda)|$, the quicker the $X(3872)$ pole moves towards the threshold, where it then turns into a virtual state that is observable on the lattice only via an unnaturally large attractive scattering length, as long as it is located very close to the threshold. If this scenario were realised in nature, it would mean that the chiral dynamics of the X is dominated by short-range mechanisms as follows from matching with the results of the pure contact theory depicted by the dashed black curve: only in the region of small pion masses $\xi \lesssim 0.9$ do the results of the pionful and pionless calculations start to deviate from each other.

On the contrary, for relatively large and positive values of $f(\Lambda)$, the short-range contributions from the OPE and $f(\Lambda)$ interfere destructively so that pion dynamics starts playing an important role. Specifically, for $(\partial E_B / \partial m_\pi)|_{m_\pi=m_\pi^{\text{ph}}} = 0.7 \times 10^{-2}$ (the upper bound of the allowed region of $f(\Lambda)$ in Fig. 4), one observes a sizeable deviation of the predictions of the theory with dynamical pions (red filled band in the left panel of Fig. 5) from the results based on the pure contact interactions (red dot-dashed curve).

Further, in the right panel of Fig. 5 we show how the results of the full dynamical theory saturate those based on the pure contact interactions. To this end we use unnaturally low cut-offs $\Lambda \in [150, 250]$ MeV, that result in integrating out pionic degrees of freedom to a

large extent, and observe, as expected, that the predictions of the full theory with dynamical pions converge to those for the pure contact theory, as formulated in Sec.4 (compare the blue hatched and the red filled bands in Fig. 5 versus the red dash-dotted curve). On the other hand, the deviation of the blue band from the dash-dotted curve can be accounted for by the fact that the pion mass dependence stemming from the OPE, as provided by the propagator $D_3(m_\pi)$ (see (7)) and the coupling constant $g(m_\pi)$ (see (23)) is more rich than just the m_π^2 -variation of the contact term, as given by (29).

Finally, it should be understood that the m_π -dependence of the $X(3872)$ mass also receives contributions from m_π -dependence of the masses of the D and D^* mesons. While these contributions can be straightforwardly calculated in the framework of chiral perturbation theory and taken into account in the analysis of $M_X(m_\pi)$, they are irrelevant for the analysis of the binding energy of the $X(3872)$, that is, the energy relative to the neutral two-body threshold $m_{*0} + m_0$. The m_π -dependence of the binding energy of the $X(3872)$ induced by the m_π -dependence of m_* and m in the kinetic energy terms entering the two- and three-body propagators is numerically small and lies beyond the accuracy of our analysis.

To summarise, the pion mass dependence of the binding energy of the $X(3872)$ analysed in our work appears to be highly nontrivial. Future lattice QCD calculations are expected to be able to distinguish between different scenarios considered in our work and might provide further insights into the chiral dynamics of the X .

6. Summary

In this work the pion mass dependence of the $X(3872)$ was studied at leading order in chiral effective field theory. To achieve cut-off independence both at the physical pion mass as well as for unphysical pion masses, it was necessary to take into account a cut-off dependence of both the m_π -independent and m_π -dependent parts of a $D\bar{D}^*$ contact interaction. Then while the former is determined by fixing the binding energy of the X at the physical point when varying the regulator, the latter involves, at the order we are working, one unknown constant. This constant was expressed in terms of the slope of the binding energy of the X at the physical point which is, in principle, measurable in lattice QCD. Depending on the employed value of the slope, different scenarios for the binding energy of the $X(3872)$ were found to be possible ranging from its quick disappearance to an increasingly deeply bound state when the pion mass is increased. Thus, once the pion mass dependence of the binding energy of the $X(3872)$ is determined in lattice QCD, the m_π -dependence of the $D\bar{D}^*$ scattering potential can be deduced from our work leading to nontrivial insights into the binding mechanism of the $X(3872)$.

At the same time, our work demonstrates that the pion mass dependence of the $X(3872)$ binding energy is nontrivial for all scenarios and provides a solid basis for chiral extrapolations of this important quantity. We further emphasise that in order to determine the m_π -dependence of the $X(3872)$ mass $M_X = m_0 + m_{*0} - E_B$, one needs to know accurately the pion mass dependence of the neutral $D\bar{D}^*$ threshold. While it can be straightforwardly calculated in the framework of chiral perturbation theory, we believe that the most accurate determination of the binding mechanism of the $X(3872)$ on the lattice would require a

simultaneous measurement of the $X(3872)$ mass and the $D\bar{D}^*$ threshold.

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